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Tau functions and the twistor theory of integrable systems

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Abstract

We first define τ-functions as generalized cross-ratios of four points on a finite- or infinite-dimensional Grassmannian. We show how this definition can be used to construct a natural flat connection on a determinant line bundle associated with two equivariant holomorphic vector bundles over a twistor space, provided that the action of the symmetries on the bundles has the same normal form at the fixed points for the two bundles. The determinant line bundle has a natural meromorphic section of which the logarithmic covariant derivative is the logarithmic derivative of the τ-function. We establish a natural product formula for this τ-function; we show that it vanishes at the jumping lines of one bundle and has poles at the jumping lines of the other. We also show that this definition leads to standard expressions for the τ-functions of the KdV equation, the Ernst equation, and the isomonodromic deformation equations. We describe a new twistor treatment of the isomonodromic deformation equations. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

Deep connections between quantum field theory and integrable systems suggest that the τ-function of Sato, Jimbo, and Miwa should play a central role in our understanding of integrability. [1] In field theory, τ has a general if sometimes heuristic definition as a correlation

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[1] The literature on τ-functions is large, and we shall not attempt to survey it in detail. The definition based on quantum field theory is given in [21–23]. We draw attention particularly to three more recent papers, [9,17,25], that look at τ from a general point of view.

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function. In integrability theory, it has been treated rigorously in certain contexts, notably by Segal and Wilson [24] in the case of solutions of the KdV equation with convergent Baker functions. Our purpose in this paper is to explain how τ-functions can be defined in a general way for integrable systems that emerge from twistor theory. Our construction incorporates that of Segal and Wilson [24] for the KdV equation, but extends it to a larger class of local solutions. It also applies to the isomonodromic deformation equations, where it gives the formula of Jimbo et al. [6], and to the Ernst equation in general relativity, where τ is identified with the conformal factor on the space of orbits. In the case of the Schlesinger equations, it leads to an alternative derivation of Malgrange's construction of τ [10].

The twistor theory of integrable systems starts from Ward's treatment of the self-dual Yang–Mills equations, in which a Yang–Mills connection D with gauge group $\mathrm{GL}(n, \mathbb{C})$ is shown to correspond to a holomorphic rank-n vector bundle E over a neighbourhood $\mathcal{Z}$ of a line in the projective space $\mathbb{CP}_3$. Neighbouring lines $X \subset \mathcal{Z}$ correspond to points in an open set in $\mathbb{C}^4$ (complex space-time), and D is recovered on this set by finding a global holomorphic trivialization of $E|_X$. To be explicit, suppose that λ, μ, z are inhomogeneous coordinates on $\mathbb{CP}_3$ and that E is specified by giving its patching matrices $P_{ij}(\lambda, \mu, z)$ relative to some open cover V_i. Given a line X determined by the linear equations $\lambda = zx + \tilde{y}$, $\mu = zy + \tilde{x}$, one first solves the Riemann–Hilbert problem of finding holomorphic maps $f_i : X \cap V_i \to \mathrm{GL}(n, \mathbb{C})$, parametrized by the space–time coordinates $x, y, \tilde{x}, \tilde{y}$, such that

$$P_{ij}(zx + \tilde{y}, zy + \tilde{x}, z) = f_i^{-1} f_j.$$

The f_is are then solutions to the linear system of the self-dual Yang–Mills connection,

$$\mathrm{D}_x f - z\mathrm{D}_{\tilde{y}} f = 0 = \mathrm{D}_y f - z\mathrm{D}_{\tilde{x}} f,$$

from which D itself can be found (up to gauge freedom, which appears here as the freedom in the choice of the 'splitting matrices' f_i). If the Riemann–Hilbert problem can be solved for one choice of X, then D will be meromorphic on the subset of space–time for which the corresponding lines X are contained in $\mathcal{Z}$, with singularities at the points for which $E|_X$ is nontrivial as a holomorphic bundle.

The construction can be generalized to obtain other systems of equations by taking $\mathcal{Z}$ to be some other complex manifold, and by replacing the projective lines by some other family of embedded copies of the Riemann sphere (the 'twistor lines'). A large proportion of the known examples of integrable systems can be obtained by imposing symmetry either directly on Ward's original construction or on one of these generalizations [27]. Those that arise directly are the symmetry reductions of the self-dual Yang–Mills equations, which include the Korteweg–de Vries and nonlinear Schrödinger equations [12]. Those that arise from a generalization include the Schlesinger equations and the deformation equations, which determine the isomonodromic deformations of an ordinary differential equation (see Appendix A). Some of the general results in this area are surveyed in [15].

Our aim is to give a general geometric construction for the τ-functions of the systems that arise from imposing symmetry under a Lie algebra action on a twistor space $\mathcal{Z}$, where the dimension of the Lie algebra is one less than the dimension of $\mathcal{Z}$ (this framework includes all the examples just mentioned). Thus we seek to associate τ-functions with equivariant holomorphic vector bundles $E \to \mathcal{Z}$. We shall see that a key part is played by the action of the Lie algebra in a formal neighbourhood of the 'singular points', where either the generating vector fields of the Lie algebra fail to be independent or the twistor lines are tangent to the orbits of the Lie algebra.

We begin with the definition of τ-functions in terms of the geometry of finite- and infinite-dimensional Grassmannians. Here we develop the point of view that τ is an invariant (which generalizes the cross-ratio) of *four* points of a Grassmannian. An important special case of this involves fixing all but one of the arguments of τ. When the remaining point moves under the action of a subgroup Γ of the general linear group, τ becomes a function on Γ that measures the failure of a canonical holomorphic section of the determinant bundle to be invariant. This framework gives a definition — essentially the Segal–Wilson definition — in terms of the geometry of the infinite Grassmannian, within which τ can be defined for a certain class of solutions of equations of KdV type. In the case of the KdV equation:

$$4v_t = 6vv_x + v_{xxx}, \tag{1}$$

τ is a logarithmic potential for v,

$$v = -\tfrac{1}{2}\partial_{xx} \log \tau. \tag{2}$$

In the twistor construction, the solutions of the KdV equation correspond to a class of $SL(2, \mathbb{C})$ holomorphic vector bundles over an open set (a neighbourhood of a section) in the tangent bundle of the Riemann sphere. The bundles are characterized by two conditions. First, that they should be equivariant along the flow of a certain vector field L tangent to the fibres, and second that the linear action of this vector field should take a standard form at its zeros. In standard coordinates (z on $\mathbb{CP}_1$ and ζ on the fibres):

$$L = \partial_\zeta,$$

which has a second-order zero at $z = \infty$. The equivariance condition is expressed by the existence of a linear first-order differential operator $\mathcal{L}_L$ on the sections of the bundle (essentially the lift of L to the total space). Its standard form is

$$\mathcal{L}_L = \partial_\zeta - z^{-1}\begin{pmatrix} 0 & z \\ 1 & 0 \end{pmatrix} + O(z^{-k})$$

as $z \to \infty$. One can choose k in the error term as large as one pleases by using the iterative procedure of Drinfel'd and Sokolov [2] to change the trivialization of the bundle in successive formal neighbourhoods of the point at infinity; but only in the case that the Baker function converges can $\mathcal{L}_L$ be reduced exactly to the normal form.

In seeking to extend the definition of τ to include more general solutions of the KdV equation and to other integrable systems, we are led to define τ as a holomorphic function on the space M of twistor lines in $\mathcal{Z}$ associated with two equivariant bundles E and E' for which the symmetry action of the Lie algebra has the same form in some formal neighbourhood of the singular points.

We shall make this definition precise, and prove two key properties of τ in this general setting: first that it is meromorphic on M with zeros at the jumping lines of E and poles at the jumping lines of E'; second that it has a cocycle property $\tau(E, E')\tau(E', E'')\tau(E'', E) = 1$, which reflects one of the basic properties of the 'generalized cross-ratio'. We shall also show how τ relates to the standard τ function in three central examples: the KdV equation, the Ernst equation, and the isomonodromic deformation equations (the general twistor description of the latter is new, and is described in Appendix A).

The paper is organized as follows. The next section describes the definition of τ as a generalized cross-ratio of four points in a Grassmannian Gr, first for a finite-dimensional vector space, and then for a general Hilbert space V. It draws heavily on techniques from [18]. Section 3 explains its connection with a standard symplectic structure on Gr $\times$ Gr, and Section 4 explains a 'parametrized version' of the construction, in which τ is used to define a connection on a determinant line bundle when V is replaced by a vector bundle $V \to M$, with distinguished sub-bundles in place of points of Gr. In Section 5, this construction is applied to a pair E, E' of equivariant holomorphic vector bundles over a twistor space $\mathcal{Z}$. Here the points of M are the twistor lines and the fibres of V are the square-integrable sections of E over a family of circles surrounding the singular points on these lines. The two bundles are identified in a neighbourhood of the critical points, and the distinguished sub-bundles of V are given by the positive and negative frequency sections of E and E'. The result is a determinant line bundle det $\to M$, with a flat connection and a distinguished meromorphic section, with zeros at the jumping lines of E and poles at jumping lines of E'. Its logarithmic covariant derivative is the gradient of $\log\tau$. Section 6 describes three examples in detail. Finally, in Appendix A, we describe a new twistor treatment of the isomonodromic deformation equations which is used in the recovery of the τ-function of Jimbo et al. [6].

The holomorphic vector bundles on twistor space have both Čech and Dolbeault descriptions. The Čech description leads to the Grassmannian approach developed here, while the Dolbeault description leads to a definition based on Quillen determinants of d-bar operators, which will be addressed in another paper.

2. Tau as a generalized cross-ratio

In this section, we shall define τ a meromorphic function of four points on a Grassmannian, and obtain a formula for its gradient in terms of Toeplitz operators. When three of the four arguments are fixed, we recover the definition given by Segal–Wilson.

2.1. *Tau functions in finite dimensions*

We shall set the scene by considering the finite-dimensional case. Let V be an even ($2n$) dimensional complex vector space, and let Gr denote the Grassmannian of n-dimensional subspaces of V. A pair of $(V_+, V_-) \in \mathrm{Gr} \times \mathrm{Gr}$ is called a *polarization* whenever

$$V_+ \cap V_- = 0 \quad \text{and} \quad V_+ + V_- = V.$$

These are equivalent in finite dimensions, and either implies that $V = V_+ \oplus V_-$. The set of all polarizations will be denoted by $\mathcal{P}$.

Given three subspaces $W_1, W_2, W_3 \in \mathrm{Gr}$, with $(W_2, W_3) \in \mathcal{P}$, we have a projection $W_1 \to W_2$ along W_3. We denote this by

$$W_1 \xrightarrow{W_3} W_2 \quad \text{or by} \quad W_1 \xrightarrow{3} W_2$$

Given four subspaces $W_1, W_2, W_3, W_4 \in \mathrm{Gr}$ in general position, we define the invariant $\tau(W_1, W_2, W_3, W_4) = \tau_{1234}$ by

$$\tau(W_1, W_2, W_3, W_4) = \frac{\det(W_1 \xrightarrow{3} W_2)}{\det(W_1 \xrightarrow{4} W_2)} = \det(W_1 \xrightarrow{3} W_2 \xrightarrow{4} W_1).$$

This is independent of the choice of bases in W_1 and W_2 and is meromorphic in its four arguments. When $n = 2$, the Grassmannian is the projective line and τ is the cross-ratio. In the general case, τ shares the following properties with the cross-ratio.

Proposition 1. *For any* $W_i \in \mathrm{Gr}$,
(a) $\tau_{1234} = \tau_{3412} = \tau_{2134}^{-1}$;
(b) $\tau_{1234}\tau_{1245}\tau_{1253} = 1$;
(c) $\tau_{1234} = 0$ *whenever* $W_1 \cap W_3 \neq 0$, *with the other arguments in general position;*
(d) *If* W_1 *is the graph of a linear map* $A : W_2 \to W_4$ *and* W_3 *is the graph of a linear map* $B : W_4 \to W_2$, *then* $\tau_{1234} = \det(1 - BA)$.

Proof. We note first that (b) and (c) are immediate from the definition. For (a), we need an explicit formula for τ. We write the identity on V as a map $W_1 \oplus W_3 \to W_2 \oplus W_4$ in the block form:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{3}$$

where $a : W_1 \to W_2$ is the map $W_1 \xrightarrow{4} W_2$. We can similarly write the identity on V as a map $W_1 \oplus W_4 \to W_2 \oplus W_3$ in the block form:

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix},$$

where a' is the map $W_1 \xrightarrow{3} W_2$. By considering the decomposition of $w \in W_1$ into a sum of vectors in W_2 and W_4 in two different ways, we find that

$$a' + bc' = a, \quad dc' = c,$$

and hence that $a' = a - bd^{-1}c$. Therefore,

$$\tau_{1234} = \det(1 - bd^{-1}ca^{-1}). \tag{4}$$

The first equality in (a) follows by replacing (3) by its inverse. The second equality then follows from the definition. For the final statement, note that $A = ca^{-1}$, $B = bd^{-1}$. □

The map $BA : W_2 \to W_2$ is well defined whenever (W_2, W_3) and (W_1, W_4) are polarizations: it is $w_2 \mapsto w_2'$, where one first writes $w_2 = w_1 + w_4$, and then $w_4 = w_2' + w_3$, with $w_1 \in W_1$, $w_2' \in W_2$, $w_3 \in W_3$, and $w_4 \in W_4$.

2.2. *Tau functions in Hilbert space*

We now extend these various definitions to an infinite-dimensional Hilbert space V. In the setting that we shall need for twistor theory, V will be the space of square-integrable maps from a circle on the Riemann sphere (or more generally a finite collection of circles) into $\mathbb{C}^n$.

By a *polarization* (V_+, V_-) of V we mean a pair of closed subspaces such that $V = V_+ \oplus V_-$. Two polarizations (W_1, W_3) and (W_2, W_4) are in the same *polarization class* whenever, in (3) above, a and d are zero-index Fredholm operators and b and c are Hilbert–Schmidt operators (note that all the blocks are bounded).

Remark. *As a technical aside, we note that 'in the same polarization class' is indeed an equivalence relation: the only point that is not immediately obvious is that, if a'', b'', c'', d'' are the blocks in the inverse of* (3), *and if b and c are Hilbert–Schmidt, then so are b'' and c''. However, in this case we have the identities*

$$aa'' + bc'' = 1, \quad ab'' + bd'' = 0.$$

From the first, we have that a and a'' are inverse operators modulo the Hilbert–Schmidt error-term bc''. Since Hilbert–Schmidt operators are compact, this means that a and a'' are Fredholm operators. (Similarly d and d'' are Fredholm.) In particular, we can find an operator k of finite rank such that $a + k$ is invertible. We now can prove that b'' is Hilbert–Schmidt by multiplying the second equation by $(a + k)^{-1}$:

$$b'' = -(a + k)^{-1}bd'' + (a + k)^{-1}kb''.$$

On the right, the first term is Hilbert–Schmidt because b is, while the second has finite rank, so certainly is Hilbert–Schmidt. One proves similarly that c'' is Hilbert–Schmidt.

Given these conditions, (4) still makes sense and can be taken as the definition of τ because a product of Hilbert–Schmidt operators is trace-class, and because an endomorphism of a Hilbert space has a determinant whenever it differs from the identity by an operator of trace class (see, for example, [20], p. 323). We can therefore give τ_{1234} a well-defined value for four closed subspaces $W_i \subset V$ provided that (W_2, W_3) and (W_1, W_4) are polarizations in the same class: we simply take the formula:

$$\tau_{1234} = \det(1 - BA)$$

in Proposition 1 as the definition. The rest of the proposition then holds whenever the appropriate values of τ are defined. Part (a) is proved in the same way as before, by noting that if

$$V = W_1 \oplus W_3 = W_2 \oplus W_3 = W_1 \oplus W_4 = W_2 \oplus W_4,$$

and that if (W_1, W_3) and (W_2, W_4) are in the same polarization class, then so are (W_2, W_3) and (W_1, W_4). It therefore makes sense to permute the arguments of τ as required. Part (b) follows from the fact that

$$W_1 \xrightarrow{3} W_2 \xrightarrow{4} W_1 \xrightarrow{4} W_2 \xrightarrow{5} W_1 \xrightarrow{5} W_2 \xrightarrow{3} W_1$$

is the identity on W_1, while the τs in the triple product are the determinants of the compositions of successive pairs of maps.

Example (*The Segal–Wilson construction*). The Hilbert space $V = L^2(S^1, \mathbb{C}^n)$ has a natural polarization (V_+, V_-), where V_+ and V_- are the subspaces of functions with respective Fourier series:

$$\sum_{k>0} a_k z^k \quad \text{and} \quad \sum_{k\le 0} a_k z^k. \tag{5}$$

In the Grassmannian picture of the KdV hierarchy, $n = 1$ and the data associated with a particular solution (with convergent Baker function) give a further fixed subspace $W = W_+ \subset V$. Three of the arguments of τ are these fixed spaces, and the fourth moves under the flow of the abelian group:

$$\Gamma_+ = \left\{\exp\left(\sum t_a z^a\right)\right\},$$

which acts on V by multiplication. We define the τ-function on Γ_+ by

$$\tau_W(g) = \tau(W, V_+, gV_-, V_-) \qquad g \in \Gamma_+.$$

By Proposition 1, if W is the graph of $A : V_+ \to V_-$ and g^{-1} has the block decomposition

$$g^{-1} = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix},$$

then $\tau_W(g) = \det(1 + a^{-1}bA)$ (see [24], Eq. (3.5), p. 20). The twistor construction extends the Segal–Wilson definition to the case that the Baker function is divergent.

2.3. Toeplitz operators

Let (V_+, V_-) be a polarization and let $\mathcal{P}$ denote its polarization class. The *restricted general linear group* $\mathrm{GL}(V)$ is the group of bounded invertible linear operators $g : V \to V$ that map $\mathcal{P}$ to itself. In the finite dimensions, $\mathrm{GL}(V)$ is the standard general linear group. In the example above, $\mathrm{GL}(V)$ is the identity component of the restricted general linear group of Pressley and Segal (see [18], Section 6.3); we shall make use of the fact that it contains as subgroup the loop group $\mathrm{LSL}(n, \mathbb{C})$, acting on $L^2(S^1, \mathbb{C}^n)$ as a group of multiplication operators.

A multiplication operator g on V determines a *Toeplitz operator* $T_g : V_- \to V_-$ with respect to V_+ by

$$T_g : V_- \to V_- : v \mapsto (gv)_-.$$

We shall use the same notation T_g more generally to denote the composite of an operator g with projection back into V_- along V_+. In general $T_{g_1 g_2} \neq T_{g_1} T_{g_2}$. However, if $\mathrm{GL}_+(V)$ and $\mathrm{GL}_-(V)$ denote the subgroups of $\mathrm{GL}(V)$ of transformations that map, respectively, V_+ and V_- to themselves, then

$$T_{g_+ g g_-} = T_{g_+} T_g T_{g_-} \quad \text{whenever } g_+ \in \mathrm{GL}_+(V), g_- \in \mathrm{GL}_-(V). \tag{6}$$

We also have that if $V'_+ = f^{-1} V_+$ for some $f \in \mathrm{GL}_-(V)$, then (V'_+, V_-) is also a polarization and the operators T and T' on V_- determined by the projections along V_+ and V'_+ are related by

$$T'_g = T_f^{-1} T_{fg} \tag{7}$$

for any g.

With this notation, we have the following.

Proposition 2. *Let $k \in \mathrm{GL}_+(V)$, $h \in \mathrm{GL}_-(V)$. Then*

$$\tau(h^{-1}V_+, V_+, kV_-, V_-) = \det(T_h^{-1} T_{hk} T_k^{-1}).$$

Proof. In a basis adapted to the direct sum decomposition, we have

$$k = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, \quad h^{-1} = \begin{pmatrix} p & 0 \\ q & r \end{pmatrix}, \quad hk = \begin{pmatrix} p^{-1}a & p^{-1}b \\ -r^{-1}qp^{-1}a & r^{-1}(c - qp^{-1}b) \end{pmatrix}.$$

The formula follows by noting that $h^{-1}V_+$ is the graph of $qp^{-1} : V_+ \to V_-$, that kV_- is the graph of $bc^{-1} : V_- \to V_+$, and that $T_k = c$, and so on. □

2.4. *The gradient of* $\log \tau$

In finite dimensions, the expression in Proposition 2 can be rewritten as

$$\det T_{hk} / \det T_h \det T_k,$$

and one can use this to derive simple expressions for the derivatives of τ with respect to its arguments. In a general Hilbert space, the individual determinants are generally not well defined, and we must proceed in a different way.

We shall specialize for the moment to the case that $V = L^2(S^1, \mathbb{C}^n)$ and that (V_+, V_-) is the Fourier series polarization. Let $D = \{|z| > 1 - \epsilon\}$ and $\hat{D} = \{|z| < 1 + \epsilon\}$ denote open discs covering the Riemann sphere and intersecting in an annulus Δ containing S^1.

Suppose that we are given a holomorphic map $h : D \to \mathrm{SL}(n, \mathbb{C})$ and a smooth family of holomorphic maps $P : \Delta \to \mathrm{SL}(n, \mathbb{C})$. Then

$$\tau = \tau(h^{-1}V_+, V_+, V_-, PV_-)$$

is well defined and depends smoothly on the parameters. It vanishes where $V_+ \cap PV_- \neq 0$, which we shall assume is true for almost all values of the parameters. At a generic point, hP and P have Birkhoff factorizations of the form:

$$hP = \hat{g}^{-1}g, \quad P = \hat{f}^{-1}f,$$

where $\hat{g}, \hat{f} : \hat{D} \to \mathrm{SL}(n, \mathbb{C})$ and $g, f \to \mathrm{SL}(n, \mathbb{C})$ are holomorphic in z and depend smoothly on the parameters. We have the following.

Proposition 3. *Let* d *denote the exterior derivative with respect to the parameters and* ∂ *denote the holomorphic differential with respect to* z*. Then*

$$\mathrm{d}\log\tau = \frac{1}{2\pi i}\oint \mathrm{tr}(P^{-1}\mathrm{d}P(f^{-1}\partial f - g^{-1}\partial g)).$$

Proof. First we need a lemma. Suppose that $a : D \to \mathrm{SL}(n, \mathbb{C})$ and $A : \Delta \to \mathrm{sl}(n, \mathbb{C})$ are holomorphic. Then

$$\tau(aV_+, V_+, (1 + tA)V_-, V_-) = -\frac{t}{2\pi i}\oint \mathrm{tr}(\partial a a^{-1} A) + \mathrm{O}(t^2).$$

To prove this, we note that if $v_- \in V_-$, then the images of $(1 + tA)v_-$ in V_- under the projections along V_+ and aV_+ are, respectively,

$$tT_A v_- \quad \text{and} \quad taT_{a^{-1}A}v_-,$$

where T is defined by projection into V_- along V_+. Therefore,

$$\tau(aV_+, V_+, (1 + tA)V_-, V_-) = t\ \mathrm{tr}(T_a T_{a^{-1}A} - T_A) + \mathrm{O}(t^2).$$

The operator in brackets on the right-hand side is trace class. To compute its trace, we note that the projection of $v \in V$ into V_- is

$$v_-(z) = \frac{1}{2\pi i}\oint \frac{v(z')z}{z'(z - z')}\,\mathrm{d}z'.$$

Therefore,

$$\begin{aligned}
&\tau(aV_+, V_+, (1 + tA)V_-, V_-) \\
&= -\frac{1}{4\pi^2}\oint\oint \mathrm{tr}\left(\frac{z(a(z)a^{-1}(z') - 1)A(z')}{z'(z - z')}\right)\frac{z'}{z(z' - z)}\mathrm{d}z'\,\mathrm{d}z \\
&= -\frac{1}{2\pi i}\oint \mathrm{tr}(\partial a a^{-1}A) + \mathrm{O}(t^2).
\end{aligned}$$

The proposition now follows by writing

$$\begin{aligned}
\frac{\tau(h^{-1}V_+, V_+, V_-, (P + \delta P)V_-)}{\tau(h^{-1}V_+, V_+, V_-, PV_-)} &= \tau(h^{-1}V_+, V_+, PV_-, (P + \delta P)V_-) \\
&= \tau(\hat{f}h^{-1}V_+, V_+, V_-, f(1 + P^{-1}\delta P)V_-) \\
&= \tau(fg^{-1}V_+, V_+, V_-, f(1 + P^{-1}\delta P)f^{-1}V_-)
\end{aligned}$$

since $\hat{g}V_+ = V_+$. So the proof is completed by taking $a = fg^{-1}$ and $tA = fP^{-1}\delta P f^{-1}$. □

3. The symplectic form on $\mathcal{P}$

In the finite-dimensional case, we can understand the symmetries of τ in Proposition 1 in another way in terms of the curvature of a connection ∇ on a holomorphic line bundle over $L \to \mathcal{P}$. The fibre of the line bundle at $w = (W_+, W_-)$ is

$$L_w = \bigwedge^n W_+^*$$

and parallel transport with respect to ∇ is given by taking the determinant of the projection from W_+ to a nearby subspace $W_+' \subset V$ along W_-.

Now fix a polarization (V_+, V_-) and a volume element $\nu \in \bigwedge^n V_+^*$. We can extend ν to two local sections α, β of L by taking α_w and β_w to be the respective pull-backs of ν under the projections:

$$W_+ \xrightarrow{V_-} V_+, \quad W_+ \xrightarrow{W_-} V_+.$$

Then α is covariantly constant along $\{W_+\} \times \mathrm{Gr}$ for fixed W_+, while β is covariantly constant along $\mathrm{Gr} \times \{W_-\}$ for fixed W_-. However,

$$\beta = \tau(W_+, V_+, W_-, V_-)\alpha.$$

Therefore,

$$\nabla\alpha = -\partial_+(\log\tau)\alpha, \quad \nabla\beta = \partial_-(\log\tau)\beta,$$

where $\tau(w) = \tau(W_+, V_+, W_-, V_-)$ and $\mathrm{d} = \partial_+ + \partial_-$ is the decomposition of the exterior derivative on $\mathrm{Gr} \times \mathrm{Gr}$ along the two factors. It follows that the curvature of ∇ is

$$\Omega = \mathrm{d}\partial_- \log\tau = \partial_+\partial_- \log\tau.$$

We note that Ω extends meromorphically to $\mathrm{Gr} \times \mathrm{Gr}$ with poles where $V \neq W_+ \oplus W_-$.

If we apply (b) in Proposition 1 twice, together with (a), then we find that for any six $W_i \in \mathrm{Gr}$,

$$\log\tau_{1234} = \log\tau_{1536} - \log\tau_{2536} - \log\tau_{1246}. \tag{8}$$

Consequently, if $V = V_+' \oplus V_-'$ is some other decomposition of V, then

$$\log\tau(W_+, V_+, W_-, V_-) = \log\tau(W_+, V_+', W_-, V_-') \\ - \log\tau(V_+, V_+', W_-, V_-') - \log\tau(W_+, V_+, V_-, V_-'), \tag{9}$$

from which we see that Ω is independent of the choice of V_+ and V_- — it is a natural holomorphic symplectic structure on $\mathcal{P}$.

Before turning to the infinite-dimensional extension, it is worth explaining how Ω appears in other contexts: we shall develop in another paper the connections with quantum field theory that are suggested by the second of these correspondences.

Example (*The projective line and the hyperbolic plane*). First let us take $n = 2$, and V_+ and V_- to be the subspaces of $V = \mathbb{C}^2$ spanned, respectively, by $(1, 0)$ and $(0, 1)$. If we pick basis elements $(1, w)$ and $(z, 1)$ in W_+ and W_-, then

$$\tau = 1 - wz, \quad \Omega = \frac{\mathrm{d}z \wedge \mathrm{d}w}{(1 - wz)^2}.$$

Thus, apart from a factor of i, Ω is the analytic continuation of the standard symplectic forms on the projective line ($w = -\bar{z}$) and the hyperbolic plane ($w = \bar{z}$).

Example (*Symplectic spinors*). Let V be the complexification of a real $2n$-dimensional symplectic vector space S, and let Λ denote the positive Lagrangian Grassmannian. That is, the set of positive complex structures $J : S \to S$ compatible with the symplectic structure. Then we can embed Λ (as a real manifold) in $\mathcal{P}$ by taking W_+ and W_- to be the eigenspaces of J with eigenvalues i and $-i$, respectively. The restriction of Ω to Λ (again to within a factor of i) is the symplectic form of the natural Kähler structure on Λ. This plays a central role in the construction of the metaplectic representation by geometric quantization (Kostant's theory of symplectic spinors). Each $J \in \Lambda$ determines a polarization and hence a Fock space $\mathcal{F}_J$. The ground states in $\mathcal{F}_J$ form a line bundle over Λ (the 'half-form' bundle), which carries a natural connection with curvature equal to half the Kähler form. The same thing happens in bosonic quantum field theory, where V is an infinite-dimensional Hilbert space and the fibres of the line bundle are the vacuum states of the polarizations, and also in fermionic quantum field theory, where the J is a polarization of a real vector space with a symmetric bilinear form, F_J is the fermion Fock space, and the sign of Ω is reversed. (See [18], some of the underlying ideas are described in [30]).

3.1. The determinant bundle

Our strategy for defining τ in an infinite-dimensional setting will be to identify its logarithmic derivative with the potential 1-form of a flat holomorphic connection on an appropriate line bundle. Although L itself is not well defined when n is infinite, we can still make sense of $L \otimes \bigwedge^n V_+$, where V_+ is some fixed element of the Grassmannian, by identifying this tensor product with the *determinant line bundle*, det.

In finite dimensions, we define $\det \to \mathrm{Gr} \times \mathrm{Gr}$ by taking its fibre at $w = (W_+, W_-)$ to be

$$\det_w = (\textstyle\bigwedge(W_+ \cap W_-))^* \otimes \bigwedge(V/(W_+ + W_-)),$$

where $\bigwedge$ denotes the top exterior power. Provided that $V_+ \cap W_- = 0$, the exact sequence

$$0 \to W_+ \cap W_- \to W_+ \xrightarrow{W_-} V_+ \to V/(W_+ + W_-) \to 0$$

gives an isomorphism

$$\det = \textstyle\bigwedge W_+^* \otimes \bigwedge V_+ \tag{10}$$

(for example, see [3], Lemma 5.2.2). This both determines the holomorphic structure of det and identifies it with L on the complement of the set $W_- \cap V_+ \neq 0$.

When $w \in \mathcal{P}$, we have $\det_w = \mathbb{C}$ and there is a canonical element given by $s = 1$; this extends holomorphically to the whole of Gr × Gr, with $s = 0$ where $W_+ \cap W_- = 0$. Given the choice of V_+, the isomorphism identifies s with $\beta \otimes \nu^*$. It also allows us to transfer ∇ to the det $|_{\mathcal{P}}$. We then have the following.

Proposition 4. $\nabla s = \partial_-(\log \tau)s$.

Note that when we restrict det to a submanifold on which W_+ is fixed, this becomes

$$\nabla s = \mathrm{d}(\log \tau)s.$$

If we know ∇, then we can recover τ.

Now we turn to the definitions in infinite dimensions. Let W_+, W_- be closed subspaces of the Hilbert space V. We say that the pair (W_+, W_-) has *finite index* whenever $W_+ \cap W_-$ and $V/(W_+ + W_-)$ are both finite-dimensional, in which case we define the *index* by

$$\mathrm{ind}\,(W_+, W_-) = \dim W_+ \cap W_- - \dim V/(W_+ + W_-),$$

and the *determinant line* by

$$\det(W_+, W_-) = \bigwedge(W_+ \cap W_-)^* \otimes \bigwedge(V/(W_+ + W_-)).$$

When $W_+ \cap W_- = 0$ and $\mathrm{ind}\,(W_+, W_-) = 0$, we have $\det(W_+, W_-) = \mathbb{C}$ and we define $s = 1$. In general, $\det(W_+, W_-)$ is the determinant line of the Fredholm operator:

$$W_+ \oplus W_- \to V : (w_+, w_-) \mapsto w_+ - w_-.$$

We can use this to extend the definition of det. Let $\mathcal{P}$ denote a polarization class of direct sum decompositions of V, which we shall keep fixed for the moment. We denote by $\mathcal{W}$ the set of index-zero pairs W_+, W_- which are the images of some $(V_+, V_-) \in \mathcal{P}$ under an operator of the form:

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : V = V_+ \oplus V_- \to V = V_+ \oplus V_-,$$

where a and d are invertible, and b and c are Hilbert–Schmidt. If g is invertible, then $(W_+, W_-) \in \mathcal{P}$. In general $W_+ + W_- = \mathrm{Im} g$, while $W_+ \cap W_-$ is the image under g of

$$\ker(1 - a^{-1}bd^{-1}c) \subset V_+.$$

By fixing (V_+, V_-) and by taking $a = 1$, $d = 1$, we can identify an open subset of $\mathcal{W}$ with the corresponding set of operators b, c. As we change (V_+, V_-), the open sets cover $\mathcal{W}$ and so give it the structure of a complex Banach manifold.

The determinant lines form a holomorphic bundle $\det \to \mathcal{W}$. This we can see by again fixing (V_+, V_-) and by using the isomorphism

$$V_+ \oplus V_- \to W_+ \oplus W_- : (v_+, v_-) \mapsto (av_+ + cv_+, bv_+ + dv_-)$$

in the corresponding open set to show that the lines $\det(W_+, W_-)$ are also the determinant lines of the Fredholm operators g. Thus the restriction of det to the each open subset determined by a choice of (V_+, V_-) is a holomorphic bundle line bundle [19]. It follows also that s is a holomorphic section that vanishes wherever $W_+ \cap W_- \neq 0$.

We can alternatively identify $\det_w$, $w \in \mathcal{W}$, with the one-dimensional space $\det(W_+, V_+)$ which we define to be the quotient of the set of pairs (W'_-, λ'), where $\lambda' \in \mathbb{C}$ and W'_- is such that (W_+, W'_-) and (V_+, W'_-) are both in $\mathcal{P}$, by the equivalence relation:

$$(W'_-, \lambda') \sim (W''_-, \lambda'') \tag{11}$$

whenever $\lambda'' = \lambda' \tau(W_+, V_+, W'_-, W''_-)$. This is the infinite-dimensional analogue of the isomorphism (10). In this description, s is the equivalence class of the pair $(V_-, 1)$. Looked at in this way, the holomorphic structure of det is more obvious; it is in fact this description that we shall use in the applications.

4. The parametrized construction

So far, we have considered τ as a function of four points in the Grassmannian of a fixed vector space V. But the situation that we shall to look at is more general: in the twistor application, V is not a fixed space, but a vector bundle $V \to M$, together with two sub-bundles W_+ and W_-. The points of M label a holomorphic family of vector bundles over the Riemann sphere, the fibres of V are the sections over some circle in $\mathbb{CP}_1$ (or more generally, over some family of circles), and W_+ and W_- are the positive and negative frequency subspaces — the spaces of sections that continue holomorphically to the inside and outside of the circle. In this setting, $\mathrm{d}\log\tau$ is identified with a connection 1-form on the corresponding determinant bundle.

To motivate this twistor construction, we shall look first at its finite-dimensional model, in which we are given a rank-$2n$ holomorphic vector bundle

$$V \to M,$$

together with two rank-n sub-bundles W_+, W_- which are transverse almost everywhere. If we are also given a holomorphic connection on V, then we can use the construction above to induce a connection on W_+ and hence on $\bigwedge W_+$ by following infinitesimal parallel transport of $w \in W_+$ by projection along W_- back into W_+. This is nonsingular wherever $W_+ \cap W_- = 0$.

When the rank is infinite, we cannot make sense of the top exterior power of W_+. If, however, we are given a third sub-bundle W'_+, then we can make sense of the difference between the connections on $\bigwedge W_+$ and $\bigwedge W'_+$ as a connection on a determinant bundle $\det \to M$, with fibres $\det(W_+, W'_+)$. It is nonsingular everywhere, provided that at every point of M, one or other of the pairs (W_+, W_-) or (W'_+, W_-) is a polarization. This will be true as well in the infinite dimensional setting, where we shall think of W'_+ as a 'reference space', and of the holomorphic structure of the vector bundle as changing from Riemann sphere to Riemann sphere by twisting W_+ in some standard way relative to W'_+. More generally, W_+ and W'_+ will be the positive frequency spaces of a pair of holomorphic bundles which are identified outside the circles, and so share a negative frequency space. Then τ measures the way in which the two bundles twist relative to each other as the parameters change.

Let us continue for the moment with the finite-dimensional case, and suppose for the moment that $V = W'_+ \oplus W_-$ everywhere. Then the isomorphism above gives us

$$\det = \bigwedge(W_+ \cap W_-)^* \otimes \bigwedge(V/(W_+ + W_-)) = \bigwedge W_+^* \otimes \bigwedge W'_+, \tag{12}$$

and we have a connection ∇ on det. It is given explicitly by the following.

Proposition 5. *Let s denote the canonical section of* det *and let P be a section of* End V *such that $W'_+ = PW_+$. Let*

$$\gamma = \operatorname{tr}(T_P'^{-1} T'_{P\Gamma} - T'_\Gamma),$$

where $\mathrm{d} + \Gamma$ *is the connection on V in some local trivialization, and for an operator g on V, $T'_g : W_- \to W_-$ is the composite of g with projection into W_- along W'_+. Then $\nabla s = \gamma s$.*

Proof. Note first that P is unique up to multiplication on the left by g such that $gW'_+ = W'_+$. By (6), γ is independent of the choice.

In a local trivialization adapted to the decomposition $V = W'_+ \oplus W_-$, we can take Γ and P to have the block decomposition

$$\Gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 \\ -A & 1 \end{pmatrix}.$$

We can take the columns of

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ -A \end{pmatrix}$$

as frames for W'_+ and W_+, respectively. We then have trivializations of the top exterior powers of W'_+ and W_+^* in which $s = 1$ and the connections on $\bigwedge W'_+$ and $\bigwedge W_+^*$ are:

$$\mathrm{d} + \operatorname{tr}(a), \qquad \mathrm{d} - \operatorname{tr}(a + Ab),$$

respectively. On the other hand, $T'_\Gamma = d$, $T'_{P\Gamma} = d - Ab$, $T'_P = 1$. The proposition follows. □

The connection form γ is singular wherever T'_P is singular; that is, at the zeros of s, where $W_+ \cap W_- \neq 0$. However, if W_+ is preserved by $\mathrm{d} + \Gamma$, then the connection itself is nonsingular at these points.

Now from its definition, $\det(W_+, W'_+) = \det(W'_+, W_+)^*$. Thus interchanging W_+ and W'_+ replaces det by its dual. The connection on det is skew symmetric with respect to this operation. We can see this from the definition or by noting that if (W_+, W_-) and

(W_+, W'_-) are both polarizations, then there is no loss of generality in choosing P in so that $PW_- = W_-$. In that case we see that

$$\gamma = \operatorname{tr}(T_\Gamma - T_{P^{-1}} T_{P^{-1}\Gamma})$$

by (7), where $T_\cdot$ acts on W_- and is determined by the projection along W_+.

In the special case that W_+ and W'_+ are both preserved by the original connection, and its curvature vanishes, ∇ is also flat and we can *define* τ as a function on M by

$$\mathrm{d}\log\tau = \operatorname{tr}(T'_{P^{-1}} T'_{P\Gamma} - T'_\Gamma).$$

When we use ∇ to identify nearby fibres of V, then W_+ and W'_+ become fixed subspaces of a fixed space V and $\tau = \tau(W'_+, W_+, W'_-, W_-)$ for some fixed choice of W'_-, which brings us back to the framework of the Segal–Wilson construction.

5. Equivariant holomorphic vector bundles

We now turn to the twistor description of integrable systems. We suppose that we are given

(1) An $\mathrm{SL}(n, \mathbb{C})$ holomorphic vector bundle E over an N-dimensional complex manifold $\mathcal{Z}$ (twistor space);
(2) An $(N-1)$-dimensional Lie algebra $\mathfrak{h}$ of holomorphic vector fields on $\mathcal{Z}$ which are independent on a dense open subset of $\mathcal{Z}$ and along which E is equivariant;
(3) A holomorphic family of embeddings $\rho_m : \mathbb{CP}_1 \to \mathcal{Z}$ labelled by $m \in M$, where M is a k-dimensional complex manifold, such that $E_m = \rho_m^* E$ is trivial on a dense open subset $M_0 \subset M$ and nontrivial on its complement $J = M - M_0$ (J is called the *jumping set*).

By saying that E is equivariant along the vector fields in the Lie algebra, we mean that the local flows of $\mathfrak{h}$ can be lifted to E. Another way to say this is that each $L \in \mathfrak{h}$ acts on the local sections of E by a 'Lie derivative' operator $\mathcal{L}_L$. In a local trivialization, the operators take the form

$$\mathcal{L}_L = L + \theta_L,$$

where θ_L has values in $\mathrm{sl}(n, \mathbb{C})$ and transforms under change of trivialization by

$$\theta_L \mapsto g^{-1}\theta_L g + g^{-1}L(g).$$

The map $L \mapsto \mathcal{L}_L$ is a representation of $\mathfrak{h}$ by first-order differential operators on the local sections.

For each $m \in M$, we denote the corresponding 'twistor line' $\rho_m(\mathbb{CP}_1)$ by X_m. The vector fields in $\mathfrak{h}$ give $(N-1)$ sections of the normal bundle of X_m and hence a section ν of its top exterior power, which is a line bundle. Its zeros are a finite collection of points a_i (depending on m), at each of which some nonzero element of the Lie algebra either has a zero or is tangent to X_m. We denote by S the subset of $\mathcal{Z}$ at which the values of the vector fields in $\mathfrak{h}$

are not independent. For simplicity, we shall assume that S is a complex hypersurface (not necessarily connected), and that it is transverse to the twistor lines. Some of the singular points a_i are given by the intersections of X_m with S; the remainder are points of tangency between X_m and the nonsingular orbits of $\mathfrak{h}$.

As m varies, so does the holomorphic vector bundle $E_m \to \mathbb{CP}_1$. We want to construct a τ-function that measures, in some sense, this variation, and which vanishes on the jumping set. The idea is to construct a holomorphic line bundle det $\to M$, the fibres of which we can identify with

$$\bigwedge H^0(\mathbb{CP}_1, E \otimes \mathcal{O}(-1))^* \otimes \bigwedge H^1(\mathbb{CP}_1, E \otimes \mathcal{O}(-1)),$$

together with a flat holomorphic connection ∇. The τ-function will then be defined by $\nabla s = \mathrm{d}(\log \tau)s$.

One way to do this is by using Dolbeault cohomology, combined with Quillen's construction of the determinant bundle. Then τ emerges as a determinant of a $\bar{\partial}$-operator. We shall explore this point of view in another paper. Here we shall work instead in the Čech framework, and build on the general theory of the preceding sections.

5.1. Notation

We need some notation to help to make the transition between this twistor framework and the Grassmannian picture we have considered so far. For the moment, we shall work locally in M. We choose an orbit Σ of $\mathfrak{h}$ such that for each m, $p = \rho_m^{-1}(\Sigma)$ is a single point distinct from the a_is. For each i, we choose a small open disc $D_i \subset \mathbb{CP}_1$ containing a_i but excluding $X \cap \Sigma$, and we choose a circle $C_i \subset D_i$ surrounding a_i. We put $D = \cup_i D_i$, and $C = \cup_i C_i$. We take $\hat{D}$ to be an open set such that D, $\hat{D}$ covers $\mathbb{CP}_1$, and such that each $D_i \cap \hat{D}$ is an annulus containing C_i but excluding a_i. We take V_m to be the Hilbert space of square-integrable sections of E_m over C. The final construction of τ does not depend on the choice of Σ, nor on the Hermitian structure needed to define the inner product on V, nor on the precise choices made for C, D, and $\hat{D}$.

Within V_m, we have a pair of closed subspaces:

(i) $W_{m,-}$ is the closure of the space of smooth sections that extend holomorphically to D;

(ii) $W_{m,+}$ is the closure of the the space of smooth sections that extend holomorphically to $\hat{D}$ and vanish on Σ.

The pair has index zero by condition (3) above. Our intention is to construct a holomorphic connection on the determinant bundle of W_+ and W_-.

5.2. The connection on V

We think of the V_ms as the fibres of a bundle of Hilbert spaces $V \to M$, with the local trivializations given by picking a frame for $E|_C$ and identifying V_m with $L^2(C, \mathbb{C}^n)$. The action of $\mathfrak{h}$ determines a holomorphic connection ∇ on V with parallel transport given by moving sections of E_m along the generators of $\mathfrak{h}$.[2] To be more explicit, let $m \in M$. A vector $X \in T_m M$ determines a section of $T\mathcal{Z}$ over $\rho_m(C)$, which we can write as a sum

$$Y_X + Z_X,$$

where at each point, Y_X is in the space spanned by the vector fields in $\mathfrak{h}$ and Z_X is tangent to $\rho_m(C)$. We think of these as the contractions with X of a holomorphic 1-form Y on M with values in $\mathrm{Map}(C, \mathfrak{h})$ and a holomorphic 1-form Z with values in $\mathrm{Vect}(C)$ (the complex vector fields on C). With this interpretation,

$$\mathrm{d}Z = [Z, Z],$$

where the bracket is the Lie bracket on $\mathrm{Vect}(C)$ and d is the exterior derivative on M. We note that Y_X and Z_X extend holomorphically to D, except for poles at the zeros of ν.

Choose a holomorphic frame $\{i_a\}$ for $E \to \mathcal{Z}$ on some open set U containing $\rho_m(\hat{D})$ for each m. This gives a trivialization of E in U, and hence an identification of V_m with $L^2(C, \mathbb{C}^n)$ for each m in which the connection on V is

$$\mathrm{d} - Z,$$

where Z acts on $V_m = L^2(C, \mathbb{C}^n)$ by differentiation. If we choose some other frame e_a on C, related to i_a by $e_b = i_a P_{ab}$, then we obtain a different identification of V_m with $L^2(C, \mathbb{C}^n)$, and the connection undergoes a gauge transformation to

$$\mathrm{d} + \Gamma, \qquad \Gamma = \theta - Z,$$

where $\theta = P^{-1}\mathrm{d}P - P^{-1}Z(P)$. Here P acts as a multiplication operator on V_m, and in this trivialization, Γ acts on on V_m by a combination of multiplication by

$$\theta = P^{-1}\mathrm{d}P - P^{-1}Z(P), \tag{13}$$

which is a 1-form on M with values in the loop algebra of $\mathrm{sl}(n, \mathbb{C})$, and differentiation along Z, which is a 1-form with values in $\mathrm{Vect}(C)$. We note the following special cases.

(i) If e_a is the pull-back of a local invariant frame for $E \to \mathcal{Z}$, then $\theta = 0$.

(ii) If e_a is the pull-back of some other local frame for $E \to \mathcal{Z}$, then

$$\theta = \theta_Y,$$

where $\mathcal{L}_L = L + \theta_L$, $L \in \mathfrak{h}$, is the corresponding local form for the Lie derivative operators.

(iii) If e_a extends holomorphically to D, then θ extends meromorphically to D for each $m \in M$, with poles at the points a_i.

In considering these, one should keep in mind the different ways in which one can think about E. It is given in the first place as a bundle $E \to \mathcal{Z}$. From this, we construct a holomorphic

[2] These statements raise a number of technical difficulties in tying down the precise sense in which V is a Hilbert bundle and in treating properly the fact that Z is an unbounded operator; the difficulties are not significant, however, since the corresponding determinant bundles and operators that we construct from Z are well behaved.

family of bundles $E_m \to \mathbb{CP}_1$, labelled by $m \in M$, which we can also think of as a single bundle over the 'correspondence space' $M \times \mathbb{CP}_1$. Not every bundle over the correspondence space is the pull-back over a bundle over $\mathcal{Z}$ by the projection $(m, z) \mapsto \rho_m(z)$: it is critical to the constrcution that the families bundles over $\mathbb{CP}_1$ that we consider have this property; nor is it true that every local holomorphic frame for E_m depending holomorphically on m is the pull-back of a local frame for $E \to \mathcal{Z}$. Two general frames for E_m are related by a transition matrix $h(m, z)$, whereas for two pulled-back frames, $h = h(\rho_m(z))$ — that is, h is constant along the fibres of the projection $M \times \mathbb{CP}_1 \to \mathcal{Z}$.

The first two statements above are immediate. The third follows because it holds when e_a also has the property in (ii), and because if it holds for one frame that extends to D, then it holds for any frame that extends to D. To see this, we note that that if e_a and e'_a extend to D, then $e_b = e'_a h_{ab}$, where $h : M \times D \to \mathrm{SL}(n, \mathbb{C})$ is holomorphic. But then

$$\theta' = h^{-1}\theta h - h^{-1}Z(h) + h^{-1}\mathrm{d}h,$$

and the statement follows since the components of Z extend holomorphically to vector fields on D, except possibly for poles at the points a_i.

5.3. *The connection on* det

By its construction, ∇ is flat and preserves W_+. To transfer it to a connection ∇ on the determinant line bundle, we need a further sub-bundle $W'_+ \subset V$, transverse to W_-. We shall construct this by picking the frame e_a so that it extends holomorphically to D and by taking $W'_+ = V_+$, where

(iii) $V_{m,+}$ is the closure of the space sections of E over the C of the form $v_a e_a$, with the coefficients v_a extending holomorphically to $\hat{D}$ and vanishing on Σ.

We note that in the trivialization determined by e_a, the two spaces W_- and $W'_+ = V_+$ are fixed ($W_- = V_-$ is the space of sections that extend to D). The choice of $\{e_a\}$ will be tied down later. We shall work from the definition $\det = \det(W_+, W'_+)$ (see 11).

We then have $e_b = i_a P_{ab}$ and $W'_+ = PW_+$, where

$$P : M \times C \to \mathrm{SL}(n, \mathbb{C})$$

is the patching matrix of the family of bundles $E_m \to \mathbb{CP}_1$. Since E_m is trivial except when $m \in J$, we can find a *Birkhoff factorization* $P = \hat{f}^{-1} f$, where $\hat{f}$ and f extend, respectively, to holomorphic maps $M \times \hat{D} \to \mathrm{SL}(n, \mathbb{C})$ and $M \times D \to \mathrm{SL}(n, \mathbb{C})$, locally in M, with singularities on the jumping set.

A direct extrapolation of the finite-dimensional theory gives

$$\nabla = \mathrm{d} + \gamma; \quad \gamma = \mathrm{tr}(T_P^{-1} T_{P\Gamma} - T_\Gamma)$$

where T_g denotes the operator on W_- defined by following g by projection along V_+. The trace on the right-hand side is in fact well defined and can be found by following the calculations in [10]. We shall show that, as in the finite-dimensional model, this does indeed determine a global holomorphic connection on det, which is nonsingular on the jumping set.

Proposition 6. *For fixed $m \in M_0$, let $P = \hat{f}^{-1} f$ denote the Birkhoff factorization of P, with f, $\hat{f}$ taking values in* $\mathrm{SL}(n, \mathbb{C})$. *Then*

$$\gamma = \frac{1}{2\pi i} \oint_C \mathrm{tr}(f^{-1} \partial f P^{-1} \mathrm{d}P + \tfrac{1}{2} P^{-1} \partial P P^{-1} Z(P)),$$

where ∂ is the exterior derivative on C and d *is the exterior derivative on M.*

Proof. The Birkhoff factorization exists for m outside of the jumping set because E_m is trivial. We have for $v \in W_-$

$$(T_P^{-1} T_{P\Gamma} v - T_\Gamma v)(z) = \frac{1}{2\pi i} \oint \mathrm{tr}\left(\frac{z(f^{-1}(z) f(z') - 1)}{z'(z - z')} \Gamma' v(z') \right) \mathrm{d}z',$$

where the integral is around C and the prime on Γ' indicates that it is acting on functions of z'. Hence

$$\gamma = -\frac{1}{4\pi^2} \oint \oint \mathrm{tr}\left[\frac{z(f^{-1}(z) f(z') - 1)}{z'(z - z')} \Gamma' \left(\frac{z'}{z(z' - z)} \right) \right] \mathrm{d}z \mathrm{d}z'.$$

By reducing the z integral to a residue at $z = z'$ and by using the fact that $f^{-1}\partial f$ is trace-free,

$$\gamma = \frac{1}{2\pi i} \oint \mathrm{tr}(f^{-1} \partial f (\theta - \tfrac{1}{2} f^{-1} Z(f))). \tag{14}$$

However, because Z and $\hat{f}$ are holomorphic in $\hat{D}$,

$$\begin{aligned} \oint \mathrm{tr}(P^{-1} Z(P) P^{-1} \partial P) &= \oint \mathrm{tr}((Z(f) f^{-1} - Z(\hat{f}) \hat{f}^{-1})(\partial f f^{-1} - \partial \hat{f} \hat{f}^{-1})) \\ &= \oint \mathrm{tr}((Z(f) f^{-1} - 2Z(\hat{f}) \hat{f}^{-1}) \partial f f^{-1}) \\ &= \oint \mathrm{tr}(-f^{-1} Z(f) + 2P^{-1} Z(P)) f^{-1} \partial f). \end{aligned}$$

The proposition now follows by using (13). □

Two key properties of ∇ are given by the following propositions.

Proposition 7. *The connection ∇ is holomorphic at the jumping points.*

Proof. At points of J, s vanishes and γ is singular. We need to show that we can find a holomorphic function u in a neighbourhood of each point of J such that $u^{-1}s$ is nonvanishing and $\gamma - \mathrm{d}\log u$ is nonsingular. The first requirement is equivalent to the condition that

$$u(m) = \tau(P^{-1} V_+, V_+, V_-, V'_-) v(m)$$

for some local holomorphic function v on M and some V'_- such $(P^{-1} V_+, V'_-)$ and (V_+, V'_-) are polarizations. We note that as a consequence of (9), the ratio

$$\tau(P^{-1} V_+, V_+, V_-, V'_-) / \tau(P^{-1} V_+, V''_{m,+}, V_-, V''_{m,-})$$

is holomorphic on the jumping set for any generic choice of V''_+, V''_-, depending holomorphically on m.

The singularities in γ arise from the failure of the factorization of P for $m \in J$. However, we can choose $h: D \to \mathrm{GL}(n, \mathbb{C})$ such that hP does have a factorization $hP = \hat{g}g^{-1}$ (in fact this will be the case for a generic choice of h). We now put

$$u(m) = \tau(h^{-1}V_+, V_+, PV_-, V_-) = \tau(P^{-1}V_+, P^{-1}h^{-1}V_+, V_-, P^{-1}V_-).$$

Then, as in Proposition 3,

$$\mathrm{d}\log u = \frac{1}{2\pi i}\oint \mathrm{tr}(f^{-1}\partial f - g^{-1}\partial g)P^{-1}\mathrm{d}P,$$

and the result follows by taking $V''_+ = P^{-1}h^{-1}V_+$ and $V''_- = P^{-1}V_-$. □

Proposition 8. *The curvature of* ∇ *is the 2-form*

$$\mathrm{d}\gamma = \frac{1}{4\pi i}\oint \mathrm{tr}(\theta \wedge \partial\theta).$$

Note the appearance here of the standard cocycle on the loop algebra [18], p. 39.

Proof. First we need some notation. We put

$$\alpha = \mathrm{d}\hat{f}\hat{f}^{-1}, \quad A = \partial\hat{f}\hat{f}^{-1}, \quad \beta = \mathrm{d}ff^{-1}, \quad B = \partial ff^{-1},$$

where, as above d is the exterior derivative on M and ∂ is the exterior derivative on C (we adopt the convention that these operators commute). Then we have the structure identities

$$\mathrm{d}A = \partial\alpha - [A, \alpha], \quad \mathrm{d}\alpha = \alpha \wedge \alpha,$$

with analogous identities for β and B. Also, $P^{-1}\mathrm{d}P = f^{-1}(\beta - \alpha)f$ and

$$\mathrm{tr}(P^{-1}\mathrm{d}P \wedge \partial(P^{-1}\mathrm{d}P)) = \mathrm{tr}((\beta - \alpha) \wedge (\partial(\beta - \alpha) - [B, \beta - \alpha])).$$

The curvature is found by calculating $\mathrm{d}\gamma$ directly, term by term. First we have

$$\begin{aligned}
\mathrm{d}\oint \mathrm{tr}(f^{-1}\partial f P^{-1}\mathrm{d}P) &= \mathrm{d}\oint \mathrm{tr}(B\alpha) \\
&= \oint \mathrm{tr}(\mathrm{d}B \wedge \alpha + B\mathrm{d}\alpha) \\
&= \oint \mathrm{tr}(\partial\beta \wedge \alpha + B(\alpha \wedge \alpha - \beta \wedge \alpha - \alpha \wedge \beta) \\
&= \tfrac{1}{2}\oint \mathrm{tr}((\beta - \alpha) \wedge \partial(\beta - \alpha) + 2B(\beta - \alpha) \wedge (\beta - \alpha)) \\
&= \tfrac{1}{2}\oint \mathrm{tr}(P^{-1}\mathrm{d}P \wedge \partial(P^{-1}\mathrm{d}P)),
\end{aligned}$$

since there is no contribution to an integral over C from any term involving just A and α (which are positive frequency) or just B and β (which are negative frequency). We also have, since $P^{-1}\partial P$ and the components of the 1-form $P^{-1}Z(P)$ commute,

$$\begin{aligned} \mathrm{d}\oint \operatorname{tr}(P^{-1}Z(P)P^{-1}\partial P) &\\ = \oint \operatorname{tr}(P^{-1}[Z,Z](P)P^{-1}\partial P - 2P^{-1}Z(P)P^{-1}\partial(\mathrm{d}P)) &\\ = \oint \operatorname{tr}(P^{-1}Z(P)\wedge \partial(P^{-1}Z(P)) - 2P^{-1}Z(P)\partial(P^{-1}\mathrm{d}P)). & \end{aligned}$$

The proposition now follows from (13). □

Finally, we consider the extent to which τ depends on the choice W'_+. We suppose that instead of taking $W'_+ = V_+$, we take $W'_+ = f'^{-1}V_+$, where f' extends to a holomorphic map $f' : M \times D \to \mathrm{SL}(n, \mathbb{C})$, locally in M. With this choice we have the following.

Proposition 9. *If* $V_+ = f'W'_+$, *then the connection on* det *is* $\mathrm{d} + \gamma'$, *where*

$$\gamma' = \gamma(f, \theta) - \gamma(f', \theta),$$

with

$$\gamma(f, \theta) = \frac{1}{2\pi i}\oint \operatorname{tr}(f^{-1}\partial f(\theta - \tfrac{1}{2}f^{-1}Z(f))).$$

In particular, when $f' = 1$, this is Eq. (14).

Proof. The operators T' on $W_- = V_-$ given by projection along $f'^{-1}V_+$ are related to those given by projection along V_+ by

$$T'_g = T_{f'}^{-1}T_{f'g}$$

for any g. Hence if take $W'_+ = f'^{-1}V_+$, then we have $W'_+ = FW_+$ where $F = f'^{-1}P$ and

$$\begin{aligned} \gamma' &= \operatorname{tr}(T_F'^{-1}T'_{F\Gamma} - T'_\Gamma) \\ &= \operatorname{tr}(T_f^{-1}T_{f\Gamma} - T_{f'}^{-1}T_{f'\Gamma}) \\ &= \operatorname{tr}(T_f^{-1}T_{f\Gamma} - T_\Gamma) - \operatorname{tr}(T_{f'}^{-1}T_{f'\Gamma} - T_\Gamma). \end{aligned}$$

The result now follows in the same way as in the derivation of Eq. (14). □

Note in particular that if $f' - 1$ has a zero at each a_i of at least the same order as the pole in θ or Z, then $\gamma = \gamma'$.

6. The τ-function of a pair of equivariant bundles

What we need to do to complete the construction is to choose W'_+ so that ∇ is flat. We shall do this by taking W'_+ to be the space of positive frequency sections of a second equivariant bundle E' satisfying (1)–(3), for which the action of $\mathfrak{h}$ has the 'same normal form' on S. The result will then be a τ-function on M associated with the pair E, E'. In some cases there is a natural choice for E' as a standard 'reference bundle'. For example, for the KdV hierarchy, we can take E' to be the bundle corresponding to the trivial solution; in other cases, for example, that of the isomonodromic deformation equations, it is more natural to think of τ as a ratio of the τ-functions associated with the individual bundles.

We need to explain what we mean by the 'same normal form'. If p is a point of $\mathcal{Z}$ not on S, then we can identify E and E' equivariantly in a neighbourhood of p by choosing invariant frames for each bundle: the action of $\mathfrak{h}$ then has the same (trivial) form for both. On the other hand, if $p \in S$, then the stabilizer $\mathfrak{h}_p$ of p is nontrivial and acts linearly on E_p. When the linear action is nontrivial, there are no invariant frames in any neighbourhood of p. However, it may still be possible to choose a frame in which the $\mathfrak{h}$ takes some standard normal form. In the twistor description of integral systems, the normal form is characteristic of the system under consideration, and bundles with the same normal form correspond to solutions of the same system. We shall see in the examples that the construction of such a frame is by the standard dressing procedure in the theory of integrable systems, in which the basis vectors are developed as formal power series about the singular point. These power series do not converge in general, but by truncating the iteration after a finite number of steps, we can standardize the action of $\mathfrak{h}$ in any formal neighbourhood of S.

We do not have to consider the details of the dressing procedure to say what it means for the actions on E and E' to have the same normal form (although they *are* needed in the examples to show that the definition is not vacuous). It means that in some neighbourhood U of S there is a holomorphic section σ of $\mathrm{Hom}(E, E') \to U$ such that

$$\mathcal{L}_L \circ \sigma - \sigma \circ \mathcal{L}'_L \tag{15}$$

vanishes on S to an appropriate order for every $L \in \mathfrak{h}$. To be more precise, let X be a twistor line. Then the right-hand side of (15) determines a meromorphic section over $U \cap X$ of $\mathrm{Hom}(E, E') \otimes N^*$, where N is the normal bundle, which in general has a pole at $X \cap S$, say of order $r + 1$. We require that it should in fact be holomorphic at $S \cap X$ for every X, and should vanish to order $r + 2$. When this is satisfied, we say that E and E' are $\mathfrak{h}$ *equivalent* and we call σ an $\mathfrak{h}$-*equivalence*.

Given E, E' and σ, take $W'_{m,+} \subset V_m$ to be the closure of the the space of smooth sections v of E_m over C such that σv extends holomorphically to a section of E'_m over $\hat{D}$ and vanishes on Σ (for simplicity, we assume that $D = U \cap X$ for every twistor line; in a component D_i which does not contain a point of $X \cap S$, we construct σ by choosing local invariant frames for the two bundles). We then have a determinant bundle $\det \to M$, and a connection ∇ which is defined everywhere except on the jumping set of E' (we shall remove this restriction in a moment).

We can obtain a local formula for ∇ by picking frames. We choose a holomorphic frame e_a for E_m in D (depending holomorphically on m); we take i_a and i'_a to be the pull-backs to $\hat{D}$ of local invariant frames for E and E'. Let P and P' be the patching matrices. We then have

$$\theta = \theta' + \epsilon, \tag{16}$$

where θ and θ' have poles at the points a_i and ϵ is holomorphic at these points, with zeros of greater order than the order of the poles. If X is not a jumping line of either bundle, then $\nabla = \mathrm{d} + \gamma'$ in the trivialization determined by e_a, and, in the notation of Proposition 9,

$$\gamma' = \gamma(f, \theta) - \gamma(f', \theta) = \gamma(f, \theta) - \gamma(f', \theta'), \tag{17}$$

where $P = \hat{f}^{-1} f$ and $P' = \hat{f}'^{-1} f$ are the Birkhoff factorizations of P and P' (in the second equality, we have used (16) together with a residue calculation to replace θ by θ').

Since

$$\oint \mathrm{tr}(\theta \wedge \partial\theta) = \oint \mathrm{tr}(\theta' \wedge \partial\theta')$$

as a result of (16), we can apply Proposition 8 to show that $\mathrm{d}\gamma' = 0$ and hence that the connection is flat. We can therefore define $\tau(E, E', \sigma)$ by $\nabla s = s\, \mathrm{d}\log\tau$. As an immediate consequence of (17), we have

Proposition 10. *For any three $\mathfrak{h}$-equivalent bundles,*

$$\tau(E_1, E_2, \sigma_{12})\tau(E_2, E_3, \sigma_{23})\tau(E_3, E_1, \sigma_{31}) = 1.$$

This multiplicative property is the final version Proposition 1: it suggests that in some sense, $\tau = \det(\sigma)$.

The construction then gives us a τ-function $\tau(E, E', \sigma)$ on M, for any $\mathfrak{h}$-equivalence σ of E and E'. It is holomorphic everywhere except at the jumping points of E', and vanishes at the jumping points of E. Its definition depends only on E, E', and on the restriction of σ to the $(r+1)$th formal neighbourhood of S. As a consequence of the proposition,

$$\tau(E, E', \sigma) = \tau(E', E, \sigma^{-1})^{-1}.$$

Thus $\tau(E, E', \sigma)$ is meromorphic on the whole of M, with poles at the jumping points of E'. Its gradient $\mathrm{d}\log\tau$ is the connection 1-form (in the canonical trivialization) of a well-defined flat holomorphic connection of det, the line bundle with fibres $\det(W_+, W'_+)$; it is nonsingular everywhere on M, provided that the jumping sets of E and E' are disjoint.

7. Examples

Example (*The KdV equation*). In the construction of Mason and Sparling, a local analytic solution to the KdV equation corresponds to a holomorphic $\mathrm{SL}(2, \mathbb{C})$-vector bundle E over a neighbourhood of the zero section in $\mathcal{O}(2)$, the tangent bundle of $\mathbb{CP}_1$ [11–13]. The bundle is equivariant along the flow of the vector field:

$$L = \partial_\zeta = \tilde{z}^2 \partial_{\tilde{\zeta}},$$

where $z, \tilde{z}$ are two stereographic coordinates on $\mathbb{CP}_1$ (related by $\tilde{z} = 1/z$), and $\zeta, \tilde{\zeta}$ are the corresponding fibre coordinates (related by $\tilde{\zeta} = \zeta/z^2$). The embeddings of $\mathbb{CP}_1$ in $\mathcal{O}(2)$ are the sections $\zeta = xz + tz^2$, labelled by the space and time variables x and t, and E is required to have trivial pull-back for each x, t in some neighbourhood of $(0, 0)$.

The KdV bundles are distinguished by the form of the Lie derivative $\mathcal{L}_L$ in a neighbourhood of the fibre at infinity ($\tilde{z} = 0$), where L vanishes. They satisfy the condition that there should exist a frame in a neighbourhood of $z = \infty$ in which

$$\mathcal{L}_L = \partial_\zeta - z^{-1}\Lambda + z^{-2}a; \quad \Lambda = \begin{pmatrix} 0 & z \\ 1 & 0 \end{pmatrix},$$

where a is upper triangular at $z = \infty$. By the iterative procedure of Drinfel'd and Sokolov [2], it is then possible to change the frame so that $a = \mathrm{O}(z^{-k})$ as $z \to \infty$ for arbitrary $k > 0$.

If we use this together with an invariant frame over the complement of $z = \infty$ to construct local trivializations of $E \to \mathcal{O}(2)$, then the transition matrix $P(z, \zeta)$ satisfies

$$P^{-1}\frac{\partial P}{\partial \zeta} = -z^{-1}\Lambda + \mathrm{O}(z^{-k-2}),$$

as $z \to \infty$. We see from this that $E \sim E'$, where E' is the bundle with patching matrix $P' = \exp(-\zeta\Lambda/z)$. Although E' is trivial as a holomorphic bundle, it supports a nontrivial action of $\mathfrak{h}$.

The solution itself is recovered from P by substituting $\zeta = xz + tz^2$, and making a Birkhoff factorization $P = \hat{f}^{-1} f$ in z, with $f = 1$ at infinity. If this can be found, then

$$\begin{aligned} \partial_x \hat{f}\hat{f}^{-1} &= \partial_x f f^{-1} - f\Lambda f^{-1} + \mathrm{O}(z^{-k-1}) \\ z\partial_x \hat{f}\hat{f}^{-1} - \partial_t \hat{f}\hat{f}^{-1} &= z\partial_t f f^{-1} - \partial_x f f^{-1}, \end{aligned}$$

since $z\partial_x P = \partial_t P = z^2 \partial_\zeta P$. Both sides of the two equations must be global rational functions of z. By examining their behaviour at $z = \infty$, we conclude that the two sides of the two equations are respectively of the forms $-\Lambda + A$ and B, where A and B depend only on x, t, and A is upper triangular. Therefore,

$$\partial_x \hat{f} + A\hat{f} - \Lambda\hat{f} = 0, \quad \partial_t \hat{f} + B\hat{f} - z\partial_x \hat{f} = 0,$$

and the two operators on the left-hand sides commute; f also satisfies the second equation (but not the first in general). The integrability of this linear system implies that

$$A = \begin{pmatrix} -q & * \\ 0 & q \end{pmatrix}, \quad B = \begin{pmatrix} * & * \\ -q_x & * \end{pmatrix},$$

where $v = 2q_x$ satisfies the KdV equation $4v_t - v_{xxx} - 6vv_x = 0$. Every local analytic solution of the KdV solution arises in this way.

To calculate $\tau = \tau(E, E')$, we take the frame in which $\mathcal{L}_L$ takes the special form at infinity. We then have $Z = 0$,

$$\theta_x = -\Lambda + O(z^{-k-1}),$$
$$f^{-1}\partial_z f = -z^{-2}\int B\,dx + O(z^{-3}),$$

as $z \to \infty$, since f satisfies the second equation of the linear system (we expand f in inverse powers of z and integrate with respect to x). By a residue calculation, we obtain from Eq. (14)

$$\partial_x \log\tau = \frac{1}{2\pi i}\oint \mathrm{tr}(f^{-1}\partial f\theta_x)dz = -\int q_x dx,$$

and so $v = -\frac{1}{2}\partial_x^2 \log\tau$.

Example (*The deformation equations*). The twistor treatment of the deformation equations is described in Appendix A, and we shall use the notation introduced there. To define a τ-function, we have to pick out a codimension-one subalgebra $\mathfrak{h} \subset \mathfrak{g}$. We shall take the one given by $y_0 + y_1 = 0$, and we shall put $z = z^1/z^0$. This is equivalent to choosing the parameter on the Riemann sphere so that one pole is fixed at $z = 0$ and another at $z = \infty$. All the orbits of $\mathfrak{h}$ in $\mathcal{Z}$ are in the hypersurfaces of constant z; $\mathfrak{h}$ acts freely except on the hypersurfaces given by the vanishing of one of the z^is, the intersections of which with the twistor lines determine the poles of A. Thus S in this case is the union of the hypersurfaces $z^i = 0$, and again $Z = 0$.

The existence of frames in which the Lie derivatives take a standard form (up to a certain order) on S shows that $E \sim E'$ whenever E and E' have the same exponents of formal monodromy. If we use these frames to compute τ, then in D_i,

$$\theta = dT_i + O(z - a_i), \quad A = f_i\partial_z T_i f_i^{-1} + O(z - a_i),$$

where

$$T_i = \frac{s_i}{(z - a_i)^{r_i}} + m_i \log(z - a_i),$$

for some diagonal polynomial s_i of degree r_i and some constant diagonal matrix m_i. On doing the same for E', we have We then have

$$d\log\tau(E, E') = -\frac{1}{2\pi i}\sum_i \oint \mathrm{tr}((f_i^{-1}\partial f_i - f_i'^{-1}\partial f_i')\,dT_i),$$

so that $\tau(E, E')$ is the quotient of the τ-functions for the corresponding solutions to the deformation equations, as defined by Eq. (5.1) in Jimbo et al. [6]; see also [7,8].

Example (*The Ernst equation*). At the other extreme, we have an example in which S is empty and the singular points arise entirely from tangency between the twistor lines and the orbits of $\mathfrak{h}$. In this case, therefore, $\theta = 0$.

The Ernst equation is the reduced from of Einstein's vacuum equations for a space–time metric of the form

$$\mathrm{d}s^2 = \Omega^2 \mathrm{d}u\mathrm{d}v - J_{ij}\mathrm{d}x^i\mathrm{d}x^j,$$

where J is a symmetric 2×2 matrix such that $\det J = -r^2$, and J and Ω are functions of the two nonignorable coordinates u, v [4]. The vacuum equations come down to

$$\partial_u(rJ^{-1}\partial_v J) + \partial_v(rJ^{-1}\partial_u J) = 0, \tag{18}$$

where $r = u + v$ (which is directly equivalent to the Ernst equation) together with

$$\partial_u(\log r\Omega^2) = \tfrac{1}{4}r\,\mathrm{tr}(J^{-1}\partial_u J)^2, \quad \partial_v(\log r\Omega^2) = \tfrac{1}{4}r\,\mathrm{tr}(J^{-1}\partial_v J)^2.$$

We can deal with (18) in much the same way as the KdV equation — only the symmetry condition is different. The twistor space is an open neighbourhood in $T\mathbb{CP}_1$ (a neighbourhood of a nonzero section), but in this case E is required to be equivariant along the holomorphic vector field:

$$L = \zeta\partial_\zeta + z\partial_z.$$

Every such bundle generates a solution to (18), although further constraints on E are needed to ensure that J is real, symmetric, and has the correct determinant. These are described in [14], but will not concern us here. [3] The point of interest here is that the conformal factor Ω on the orbits is given in terms of the τ-function by $r\Omega^2 = \tau^{-1}$.

If we exclude $(0, 0)$ and $(\infty, 0)$ from $\mathcal{Z}$, then L is nonvanishing. We take M to be an open set in the u, v-plane, and put $m = (u, v)$. The twistor lines are given by

$$\rho_m(z) = (z, \zeta) = (z, u(z-1)^2 + v(z+1)^2).$$

The Lie algebra $\mathfrak{h}$ is one-dimensional and is generated by L; its orbits are transverse to the twistor lines except at $z = \pm 1$. The tangent vectors $\partial_u, \partial_v \in T_m M$ are represented by meromorphic sections of $T\mathcal{Z}$ over a twistor line given by

$$2u(z-1)\partial_\zeta \quad \text{and} \quad 2v(z+1)\partial_\zeta,$$

respectively. By writing these as combinations of L and the tangent vector

$$\partial_z + 2(u(z-1) + v(z+1))\partial_\zeta$$

to a twistor line, we obtain $Z = Z_u \mathrm{d}u + Z_v \mathrm{d}v$ where

$$Z_u = \frac{z(z-1)}{r(z+1)}\partial_z \quad \text{and} \quad Z_v = \frac{z(z+1)}{r(z-1)}\partial_z. \tag{19}$$

[3] The non-Hausdorff reduced twistor space in [14,15] is the quotient of $\mathcal{Z}$ by the flow along L. The existence of a twistor construction for this equation is a consequence of Witten's observation [29] and was pointed out by Ward [26].

We take e_a to be the pull-back of an invariant frame in a neighbourhood of these points, and i_a to be the pull-back of some other invariant frame. Then the patching matrix P satisfies $\mathrm{d}P - ZP = 0$. Consequently,

$$(\mathrm{d}f - Zf)f^{-1} = (\mathrm{d}\hat{f} - Z\hat{f})\hat{f}^{-1}$$

and therefore both sides are equal to a global rational expression R in z with simple poles at $z = \pm 1$. We put

$$R = \frac{A_u + zB_u}{z+1}\mathrm{d}u + \frac{A_v + zB_v}{z-1}\mathrm{d}v,$$

for some matrices A_u, B_u, A_v, B_v depending only on u, v.

The 1-form γ in (14) has u component:

$$\gamma_u = -\frac{1}{4\pi i}\oint \mathrm{tr}(R_u f_z f^{-1})\,\mathrm{d}z,$$

where the contour surrounds the pole at $z = -1$, since f and $\mathrm{d}f$ are holomorphic inside the contour. But

$$f_z f^{-1} = \frac{r(z+1)}{z(z-1)} Z_u(f) f^{-1} = \frac{r(z+1)}{z(z-1)}(R_u - f_u f^{-1}).$$

Therefore, by discarding terms which are holomorphic inside the contour,

$$\begin{aligned}\gamma_u &= -\frac{1}{4\pi i}\oint \mathrm{tr}\left(\frac{r(z+1)}{z(z-1)} R_u(R_u - f^{-1}f_u)\right)\mathrm{d}z\\ &= -\frac{1}{4\pi i}\oint \mathrm{tr}\left(\frac{r}{z(z^2-1)}(A_u + zB_u)^2\right)\mathrm{d}z\\ &= -\tfrac{1}{4} r\,\mathrm{tr}(A_u - B_u)^2,\end{aligned}$$

together with a similar result for γ_v.

The J-matrix itself is extracted from Γ by noting that $\mathrm{d} + R$ is flat along $z = 0$ and $z = \infty$. Hence, we have

$$A_u\mathrm{d}u + A_v\mathrm{d}v = \mathrm{d}hh^{-1}, \quad B_u\mathrm{d}u + B_v\mathrm{d}v = \mathrm{d}\hat{h}\hat{h}^{-1},$$

for some matrices h, $\hat{h}$ depending on u, v. These determine J by $J = \hat{h}^{-1}h$, which then satisfies (18) as a result of the integrability condition for $\mathrm{d}+R$ on the leaves of the distribution on $M \times \mathbb{CP}_1$ spanned by $\partial_u - Z_u$ and $\partial_v - Z_v$. However, with J defined in this way, we have

$$\gamma = -\tfrac{1}{4} r(\mathrm{tr}(J_u J^{-1})^2\mathrm{d}u + \mathrm{tr}(J_v J^{-1})^2\mathrm{d}v),$$

which establishes the relationship $r\Omega^2\tau = 1$ between τ and the conformal factor in the metric (a result suggested by the work of Breitenlohner and Maison [1]).

Appendix A. The deformation equations

The deformation equations determine the *isomonodromic deformations* of a family of ordinary differential equations

$$\frac{\mathrm{d}y}{\mathrm{d}z} = Ay, \tag{A.1}$$

where y takes values in $\mathbb{C}^n$, and A is a rational matrix-valued function of $z \in \mathbb{CP}_1$, depending holomorphically on the parameter $t \in \mathbb{C}^k$. As t varies, the *monodromy data* of the equation are required to remain constant.

In the *Fuchsian* case, A has only simple poles at finite or infinite values of z, and the condition is that the deformations — the variations with t — should preserve the monodromy representation of the fundamental group of $\mathbb{CP}_1 - \{\text{poles}\}$, up to conjugacy. In the *irregular* case, there are higher order poles, and the monodromy data are more complicated [6]. At a pole $z = a_i$, we have

$$A = A_{i,r_i}(z - a_i)^{-r_i - 1} + \mathrm{O}((z - a_i)^{-r_i}),$$

where we assume that the leading coefficients A_{i,r_i} have distinct eigenvalues (or distinct modulo the integers in the case $r_i = 0$).

For each pole, it is possible to find a matrix $g_{i,0}$ that diagonalizes A_{i,r_i}, and given $g_{i,0}$, a unique formal fundamental solution:

$$Y_i = g_i(z - a_i)^{m_i} \exp(s_i/(z - a_i)^{r_i}), \tag{A.2}$$

where g_i is a formal power series in $z - a_i$ of the form $g_{i,0} + \mathrm{O}(z - a_i)$, s_i is a diagonal matrix polynomial in z of degree r_i, and m_i is a constant diagonal matrix. In general the formal series does not converge; however, in certain 'sectors' at each pole, it is possible to find actual solutions that are asymptotic to the formal solutions at the poles within the sector (there are $2r_i$ 'sectors' at the pole a_i, defined by restricting $\arg(z - a_i)$ to an open interval of length slightly greater than π/r_i). It is the transition matrices between these actual solutions as one continues from one sector to the next at the same pole (the *Stokes' matrices*) or from a sector at one pole to a sector at another (the *connection matrices*), together with the m_is (*the exponents of formal monodromy*) that must remain unchanged. The remaining freedom in A is parametrized by the positions of the poles on the z-sphere (with one fixed at $z = \infty$) and the entries in the diagonal coefficients of s_i as a polynomial in $(z - a_i)$. We denote these variables collectively by $t \in M$.[4]

The precise details are given by Jimbo et al. [6], who reduce the problem to that of finding As with dependence on t governed by the deformation equations:

$$\mathrm{d}A - \partial_z\omega - [\omega, A] = 0\,, \tag{A.3}$$

where d denotes the exterior differentiation with respect to t and $\mathrm{d} + \omega$ is a flat connection, depending rationally on z. Given A at some point of the parameter space, one finds ω by

[4] The dependence on the coefficient of $(z - a_i)^{r_i}$ is trivial, but it is useful to keep these additional redundant parameters.

requiring that the formal solutions should satisfy the linear equation $\mathrm{d}Y + \omega Y = 0$ at each pole: differentiation of the formal solution gives the singular and constant part of ω at each a_i, and hence ω itself by appealing to rationality. Thus (A.3) is a nonlinear first-order equation for A as a function of the parameters — although it is not easy to show that it is integrable.

Two obvious, but trivial, classes of deformation are: 'gauge transformations' $\partial_z - A \to g^{-1}(\partial_z - A)g$ and coordinate transformations of z. But if the latter are to be global on $\mathbb{CP}_1$, then g must be constant and the coordinate mapping must be a Möbius transformation. The nontrivial deformations can be thought of informally as combinations of 'singular gauge transformations' by $g = g_i^{-1}\exp(D_i)g_i$ at each pole, where D_i is a diagonal matrix of the form $\sum_{j\geq -r_i} D_{i,j}(z-a_i)^j$, and transformations that move the poles, but which are not induced by global coordinate mappings. The twistor construction fits these together into a consistent global deformation.

Special cases of the deformation equations are the *Schlesinger equations*, which arise when all the poles are simple, and the only freedom is to move them around on the Riemann sphere, and the *Painlevé equations*, which arise when A takes values in $\mathrm{sl}(2,\mathbb{C})$ and has four poles in total, counted according to multiplicity (the various sub-cases are determined by the coincidence properties of the poles).

Remark. *There is one point that has been glossed over above, namely the question of gauge. The deformation equations as we have defined them are the condition that* $A\,\mathrm{d}z + \omega$ *should be a flat connection* 1*-form, subject to the rationality constraints in the dependence of* A *and* ω *on* z. *We have a natural invariance, therefore under gauge transformations of the connection by* $g(t) \in \mathrm{GL}(n,\mathbb{C})$, *where* g *is independent of* z *(such transformations give trivial deformations of* A.1*). Jimbo et al. work in a fixed gauge in which the leading coefficient in the Laurent expansion of* A *at one pole (at* $z = \infty$*) is fixed. They then add propagation equations for the matrices* g_{i0} *that diagonalize the leading coefficients at the other poles. We prefer not to fix the gauge, nor to constrain the coordinate* z *so that it takes the value* ∞ *at one pole. Although more natural from a geometric point of view, our approach requires minor qualification of certain statements: for example, the Schlesinger equations do not have this gauge invariance, so one must add 'in a suitable gauge' to the statement above.*

A.1. Solutions from equivariant bundles

There are many ways to obtain solutions to the deformation equations from twistor constructions by variants of an idea of Hitchin's [5]. What is needed is a twistor space (a complex manifold $\mathcal{Z}$, together with a family of embeddings $\rho_t : \mathbb{CP}_1 \to \mathcal{Z}$), a complex Lie algebra $\mathfrak{g}$ of the same dimension N as $\mathcal{Z}$, an action of $\mathfrak{g}$ on $\mathcal{Z}$ which is free on a dense open subset $\mathcal{Z}_0$, and an equivariant rank-n holomorphic bundle $E \to \mathcal{Z}$, which is trivial on some of the lines $X_t = \rho_t(\mathbb{CP}_1)$. The action of $\mathfrak{g}$ determines a flat connection D on $E|_{\mathcal{Z}_t}$, and the restriction of D to a line X_t is meromorphic. If $E_t = E_{X_t}$ is trivial, then, in a global trivialization, the covariantly constant sections are given by an ordinary differential equation of the form (A.1), with the poles of A given by the zeros on X_t of $\Delta = L_1 \wedge \cdots \wedge L_N$, where the Ls are the generators of $\mathfrak{g}$. The order $r_i + 1$ of a pole cannot exceed the order of the corresponding zero of Δ, but is generally less in the construction below. As t varies, we obtain a solution to (A.3), with singularities at the jumping lines (and at the points where Δ vanishes identically on X_t) — the rationality of A and ω is forced by the global complex geometry of $\mathbb{CP}_1$.

The generality of this construction explains the ubiquity of the deformation equations: whenever an integrable system arises from a twistor construction, one can obtain a symmetry reduction to the deformation equations by imposing additional symmetry on the twistor space so that the dimension of the whole symmetry group equals that of twistor space itself. This is a general form of the *Painlevé test* for integrability.

A.2. A twistor construction

We shall now describe one such construction, which generates the general solution with $N + 1 \geq 2$ poles of orders $r_0 + 1, r_1 + 1, \ldots, r_N + 1$ (it also covers the case of only one pole — for example the first and second Painlevé equations — by treating a regular point of (A.1) as a 'pole' at which the singular part of A vanishes).

We denote by $\mathfrak{t}$ the abelian Lie algebra of complex $n \times n$ diagonal matrices. The twistor space $\mathcal{Z}$ is an open subset of the total space of the bundle:

$$\bigoplus \mathfrak{t}(r_i) \to \mathbb{CP}_N,$$

where $\mathfrak{t}(r)$ denotes the trivial bundle $\mathfrak{t} \times \mathbb{CP}_N$, twisted by $\mathcal{O}(r)$. This is the quotient of

$$(\mathbb{C}^{N+1} - \{0\}) \oplus \mathfrak{t} \oplus \cdots \oplus \mathfrak{t}$$

by the equivalence relation:

$$(z^0, \ldots, z^N, s^1, \ldots, s^N) \sim (\alpha z^0, \ldots, \alpha z^N, \alpha^{r_0} s^0, \ldots, \alpha^{r_N} s^N),$$

$\alpha \neq 0 \in \mathbb{C}$, $z^a \in \mathbb{C}$, $s^a \in \mathfrak{t}$. The Lie algebra $\mathfrak{g}$ is the semi-direct sum of the diagonal subalgebra of $\mathrm{sl}(N + 1, \mathbb{C})$ and $N + 1$ copies of $\mathfrak{t}$ (more generally, one would take copies of a Cartan subalgebra of the complexified Lie algebra of the gauge group). It acts on $\mathcal{Z}$ by

$$\begin{pmatrix} z^0 \\ z^1 \\ \vdots \\ z^N \end{pmatrix} \mapsto \begin{pmatrix} y_0 & & & \\ & y_1 & & \\ & & \ddots & \\ & & & y_N \end{pmatrix} \begin{pmatrix} z^0 \\ z^1 \\ \vdots \\ z^N \end{pmatrix}$$

$$\begin{pmatrix} s^0 \\ s^1 \\ \vdots \\ s^N \end{pmatrix} \mapsto \begin{pmatrix} r_0 y_0 & & & \\ & r_1 y_1 & & \\ & & \ddots & \\ & & & r_N y_N \end{pmatrix} \begin{pmatrix} s^0 \\ s^1 \\ \vdots \\ s^N \end{pmatrix} + \begin{pmatrix} (z^0)^{r_0} h_0 \\ (z^1)^{r_1} h_1 \\ \vdots \\ (z^N)^{r_N} h_N \end{pmatrix},$$

where the y_is, with $\sum y = 0$, are the entries in a diagonal element of $\mathrm{sl}(N+1, \mathbb{C})$ and the hs are elements of $\mathfrak{t}$. Note that the dimension of $\mathfrak{g}$ is $N+(N+1)n$, which is the same as that of $\mathcal{Z}$.

The embeddings $\rho_t : \mathbb{CP}_1 \to \mathcal{Z}$ are given by choosing (i) a line $X \subset \mathbb{CP}_N$ and (ii) a holomorphic section of the restriction to X of the bundle $\mathcal{Z} \to \mathbb{CP}_N$. The intersections of X with the hyperplanes $z^i = 0$ give the locations $a_i \in \mathbb{CP}_1$ of the poles, and the section itself is a collection of $N+1$ polynomials in z with values in $\mathfrak{t}$, of orders $r_0, \dots, r_N$. These are the polynomials s_i. (We have to assume that $\mathcal{Z}$ is large enough to contain $\rho_t(\mathbb{CP}_1)$ for an open subset of the parameter space.)

In explicit terms, A can be found by solving a Riemann–Hilbert problem involving the patching matrices of E_t (the restriction of E to X_t). We cover $\mathcal{Z}$ by open sets V_i on each of which E is trivial and denote by P_{ij} the corresponding patching matrices. On V_i, the action of $\mathfrak{g}$ is given by

$$\mathcal{L}_L = L + \theta_{iL}, \qquad L \in \mathfrak{g},$$

where $L(P_{ij}) = \theta_{iL} P_{ij} - P_{ij}\theta_{jL}$ (the existence of the family of matrices θ_{iL}, holomorphic on V_i, is equivalent to the equivariance of E).

Suppose that X_t is a line such that E_t is trivial. Then E_t is trivial also in a neighbourhood of t, and so we have $P_{ij} = f_i^{-1} f_j$, where the f_is are holomorphic $\mathrm{GL}(n, \mathbb{C})$-valued functions of z, t, with $z \in V_i \cap X_t$ and t in the neighbourhood of the given point.

Let T be the tangent vector to X_t such that $T(z) = z$. Then $T = \sum c_L L$ where the sum is over a basis of $\mathfrak{g}$ and the coefficients c_L are well defined at the points where the action is free (i.e. at the points at which the generators of $\mathfrak{g}$ span the tangent space to $\mathcal{Z}$). They are global rational functions on $\mathbb{CP}_1$: at points where the generators are dependent, they can have poles up to the order of the zero of $L_1 \wedge \cdots \wedge L_N$.

If we differentiate the relation $P_{ij} = f_i^{-1} f_j$ along T, then we obtain

$$\begin{aligned} z^{-1}T(P_{ij}) &= \sum c_L \theta_{iL} P_{ij} - \sum c_L P_{ij}\theta_{jL} \\ &= -f_i^{-1}\frac{\mathrm{d}f_i}{\mathrm{d}z} f_i^{-1} f_j + f_i^{-1}\frac{\mathrm{d}f_j}{\mathrm{d}z}, \end{aligned}$$

and therefore,

$$\sum f_i c_L \theta_{iL} f_i^{-1} + \frac{\mathrm{d}f_i}{\mathrm{d}z} f_i^{-1} = \sum f_j c_L \theta_{jL} f_j^{-1} + \frac{\mathrm{d}f_j}{\mathrm{d}z} f_j^{-1}. \tag{A.4}$$

Since this holds for any i, j, both sides must be equal to a global rational function of z (for fixed t), which is the matrix A.

The f_is play the role of gauge transformations between the local expression for A in terms of the Lie derivatives of the generators and the global expression in which it is a rational function of z.

A.3. Regular and irregular singularities

In this twistor construction,

$$\Delta = \left(\prod_0^N (z^i)^{r_i+1}\right)^n$$

and the poles are given by the vanishing of the z^is. The way in which the c_Ls depend on z imply that the corresponding poles in A have orders $r_i + 1$ (that is, Poincaré rank r_i).

A pole at which $r_i = 0$ is a *regular* singularity; otherwise a_i is an *irregular* singularity. Our construction of a τ-function requires that the θ_Ls should be reduced to their diagonal normal form in any formal neighbourhood of a singularity, and so we need to consider the conditions under which this is possible.

Suppose, first, that that a_0 is a regular singularity. Then the subalgebra $\mathfrak{g}_0$ given by $y_0 = 0$ acts freely in a neighbourhood of the singular point, and we can find a local holomorphic frame for E which is invariant under its action. In this frame, the θs vanish for all the generators $\mathfrak{g}_0$, so the problem is to diagonalize θ_L for some nonzero element of $\mathfrak{g} - \mathfrak{g}_0$, for example, the generator L given by $y_0 = N$, $y_1 = \cdots y_N = -1$. Now a local section invariant under g_0 is a $\mathbb{C}^n$-valued function s of $\zeta = z^0/(z^1 \cdots z^N)^{1/N}$, and L acts on such sections by

$$\mathcal{L}_L s = (N+1)\zeta \frac{\mathrm{d}s}{\mathrm{d}\zeta} + \theta_L s.$$

The singularity at a_0 is given by $\zeta = 0$; under change of invariant frame, θ_L at $\zeta = 0$ transforms by conjugation. Thus its eigenvalues are invariants. We make the standard 'genericity' assumption that they are distinct modulo the integers. We can then solve the eigensection equation $\mathcal{L}_L e = \mu e$ by developing e and μ as formal power series in powers of ζ and solving iteratively. By truncating the process at a fixed power of ζ, we can find a basis of eigensections in which θ_L is diagonal in any prescribed formal neighbourhood of $\zeta = 0$.

Now suppose, instead, that a_0 is irregular. This time we take $\mathfrak{g}_0$ to be the subalgebra given by $h_0 = 0$, and write $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{t}$, where the action of $h \in \mathfrak{t}$ is

$$s^0 \mapsto (z^0)^{r_0} h\,,$$

with the other s^is and all the z^is fixed. The Lie algebra $\mathfrak{g}_0$ acts freely in a neighbourhood of $z^0 = 0$, so we can find an invariant frame in which the corresponding θ_Ls vanish; the Lie algebra $\mathfrak{t}$ is abelian and acts linearly on the fibres of E over the hyperplane $z^0 = 0$. If we make the genericity assumption that the weights are distinct, then we can find eigensections as formal power series in z^0 by iteration, and a hence a frame (invariant under $\mathfrak{g}_0$) in which θ_L, $L \in \mathfrak{t}$, is diagonal in any prescribed formal neighbourhood of the hyperplane.

A.4. The inverse construction

Every solution to the deformation equations arises from this construction. To see this, we turn to a more geometric representation of the parameter space: we think of the polynomial s_i as a section of the bundle $\mathfrak{t}(r_i) \to \mathbb{CP}_1$, and of the positions of the poles on the Riemann sphere as being given by sections α of $\mathcal{O}(1) \to \mathbb{CP}_1$ (with the poles at the zeros, so that the pole a_i is associated with $\alpha_i = z - a_i$). Then M becomes an open subset of[5]

$$\Gamma(\mathcal{O}(1)) \oplus \cdots \oplus \Gamma(\mathcal{O}(1)) \oplus \Gamma(\mathfrak{t}(r_0)) \oplus \cdots \oplus \Gamma(\mathfrak{t}(r_N)).$$

[5] The new parameter space is one dimension higher than the original because we now have the freedom to scale the sections s_i and α_i, but the dependence on the additional parameter is trivial.

The formal solutions at the poles are of the form $g_i\alpha_i^{m_i}\exp(s_i/(\alpha_i)^{r_i})$, and a given solution to the deformation equation is a flat meromorphic connection D on a holomorphic vector bundle over $\mathbb{CP}_1 \times M$.

There is a natural projection $\pi : \mathbb{CP}_1 \times M \to \mathcal{Z}$, given by evaluating the α_is and s_is at $z \in \mathbb{CP}_1$; hence a natural family of embeddings $\mathbb{CP}_1 \to \mathcal{Z}$, labelled by the points of M. Since $s_i/\alpha_i^{r_i}$ is constant on the surfaces $\pi^{-1}(z)$, $z \in \mathcal{Z}$, and since the the formal solutions satisfy $DY = 0$, the restriction of D to these surfaces is nonsingular everywhere. Thus we can define $E \to \mathcal{Z}$ by taking the fibre over $z \in \mathcal{Z}$ to be the space of covariantly constant sections over $\pi^{-1}(z)$. This is equivariant under the action of $\mathfrak{g}$ since the effect of the action is (i) to rescale s_i and α_i in such a way as to leave the ratio $s_i/(\alpha_i)^{r_i}$ invariant, and (ii) to add a multiple of $(\alpha_i)^{r_i}$ to s_i, leaving unchanged the form of the formal solution. Following back through the construction, D is the same as the connection form $A\,\mathrm{d}z + \omega$ determined by the action of $\mathfrak{g}$ on the pull-back of E to $M \times \mathbb{CP}_1$. Thus E generates the given solution.

The columns of g_i give independent formal power series sections of E in a neighbourhood of $z^i = 0$; if these series converged, then in the corresponding frame we would have for each $L \in \mathfrak{g}$, $\mathcal{L}_L = L + \theta_L$, with

$$\theta_L = \Lambda_L := \exp(m_i \log z^i + s_i/(z^i)^{r_i})L\exp(-m_i \log z^i - s_i/(z^i)^{r_i}).$$

This is a normal form for $\mathcal{L}_L$ that depends only on the exponents of formal monodromy m_i, but not on any other other data of the solution. By truncating the formal power series, we can find a frame in which $\theta_L - \Lambda_L$ vanishes to arbitrary order on $z^i = 0$.

An immediate consequence of the existence of this twistor construction is that, apart from fixed singularities where the a_is come into coincidence, the only singularities in A and ω are algebraic singularities which arise at jumping lines (lines on which E_t fails to be trivial); [27]. Thus the truth of Conjecture 1 of Jimbo et al. [6] (p. 311) is immediate (the conjecture was proved in another way by Miwa [16] in the general case, and by Malgrange [10] in the case of the Schlesinger equation).

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